Problems involving characters and two primes

Carolina Vallejo Rodríguez

Universidad Carlos III de Madrid - Instituto de Ciencias Matemáticas de Madrid

Algebra Seminar, University of Machester

G finite group, $\mathrm{Irr}(G)$ irreducible complex characters of G.

$\mathrm{Lin}(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi(1) = 1\} \cong G/G'$, $G' = [G, G]$.

Facts (degrees)

- $\chi(1)$ divides $|G|$ for $\chi \in \mathrm{Irr}(G)$.
- $\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|$.

For a prime p, we write $\mathrm{Irr}_{p'}(G) = \{\chi \in \mathrm{Irr}(G) \mid (\chi(1), p) = 1\} \supseteq \mathrm{Lin}(G) \ni \mathbf{1}_G$.

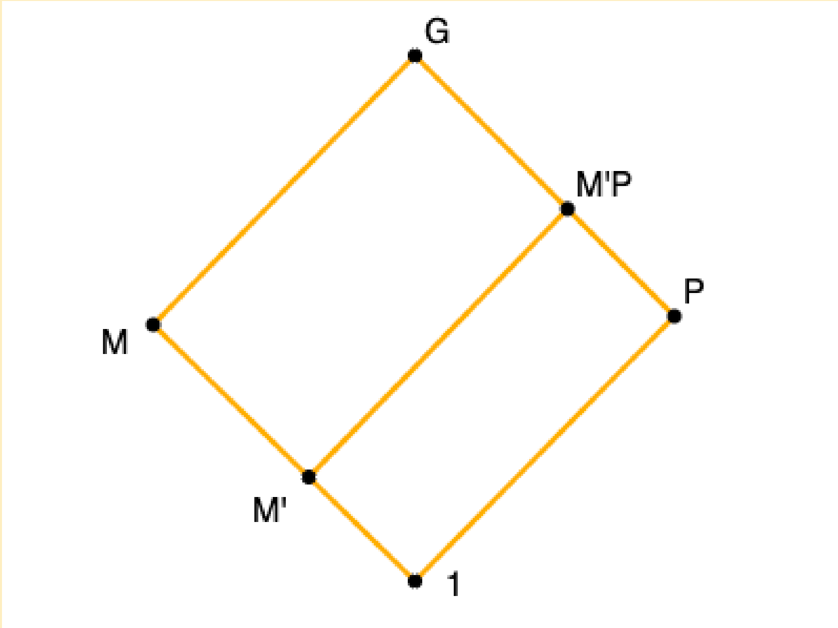
An elementary fact

- $\mathrm{Irr}_{p'}(G) = \{\mathbf{1}_G\}$ if, and only if, $G = 1$.

Proof. $\mathrm{Irr}_{p'}(G) = \{\mathbf{1}_G\}$ implies that $|G| = 1 + p^2 m$ then $(|G|, p) = 1$ and $\{\mathbf{1}_G\} = \mathrm{Irr}_{p'}(G) = \mathrm{Irr}(G)$.

Theorem (Thompson '70, Gow-Humphreys '75)

$$\mathrm{Irr}_{p'}(G) = \mathrm{Lin}(G) \iff G = M \rtimes P, \quad P \in \mathrm{Syl}_p(G), \quad \mathbf{C}_P(M') = 1.$$



By Tate's transfer theory the theorem above can be restated.

Theorem (Thompson '70, Gow-Humphreys '75)

$$\mathrm{Irr}_{p'}(G) = \mathrm{Lin}(G) \iff \mathbf{N}_G(P) \cap G' = P', \quad P \in \mathrm{Syl}_p(G).$$

Theorem (Thompson '70, Gow-Humphreys '75)

$$\mathrm{Irr}_{p'}(G) = \mathrm{Lin}(G) \iff \mathbf{N}_G(P) \cap G' = P', \quad P \in \mathrm{Syl}_p(G).$$

From now on $\pi = \{p, q\}$ and

$$\mathrm{Irr}_{\pi'}(G) = \mathrm{Irr}_{p'}(G) \cap \mathrm{Irr}_{q'}(G) \supseteq \mathrm{Lin}(G)\,.$$

Objective. Characterize $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$ for all finite groups.

Theorem (Navarro-Wolf '02)

G solvable, $\pi = \{p, q\}$ and $H \in \mathrm{Hall}_\pi(G)$.

$$\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G) \iff \mathbf{N}_G(H) \cap G' = H', \quad H \in \mathrm{Hall}_\pi(G).$$

Example. $\mathrm{Irr}_{\{2,5\}'}(\mathrm{S}_5) = \mathrm{Lin}(\mathrm{S}_5)$ but $\mathrm{Hall}_{\{2,5\}}(\mathrm{S}_5) = \varnothing$.

Objective. Characterize $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$ for all finite groups.

We first analyze the much stronger condition $\mathrm{Irr}_{\pi'}(G) = \{\mathbf{1}_G\}$.

Theorem A (Giannelli-Schaeffer Fry-V. '19)

G group and $\pi = \{p, q\}$.

$$\mathrm{Irr}_{\pi'}(G) = \{\mathbf{1}_G\} \text{ if, and only if, } G = 1\,.$$

Remark. Unlike the case $|\pi| = 1$, our proof relies on the Classification of Finite Simple Groups.

Objective. Characterize $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$ for all finite groups.

Theorem B (Giannelli-Schaeffer Fry-V. '19)

G group, $\pi = \{p, q\}$, M the solvable residual of G and $H/M \in \mathrm{Hall}_\pi(G/M)$.

$$\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G) \iff \begin{cases} \mathbf{N}_G(H)/M \cap G'/M = H'/M, \text{ and} \\ H \text{ acts on } \mathrm{Irr}_{\pi'}(M) \text{ with fixed points } \{\mathbf{1}_M\}. \end{cases}$$

- M is the smallest normal subgroup of G with solvable quotient.
- M is perfect ($M' = M$).

Example. Recall that $\mathrm{Irr}_{\{2,5\}'}(\mathsf{S}_5) = \mathrm{Lin}(\mathsf{S}_5)$. The solvable residual of S_5 is A_5.
As $H = \mathsf{S}_5$ the first part is trivially satisfied. Note that $\mathrm{Irr}_{\{2,5\}'}(\mathsf{A}_5) = \{\mathbf{1}_{\mathsf{A}_5}, \varphi, \varphi^{(1,2)}\}$.

Fact. Under the equivalent hypotheses of Theorem B, if G is nontrivial then $M < G$.
Proof. If $M = G$, then in both cases $\mathrm{Irr}_{\pi'}(G) = \{\mathbf{1}_G\}$ and $G = 1$ (by Theorem A).

2. Character degrees, fields of values and two primes

For $\chi \in \mathrm{Irr}(G)$, $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) \mid g \in G) \subseteq \mathbb{Q}(e^{2\pi i/|G|})$.

If $\mathbb{Q}(\chi) = \mathbb{Q}$, say χ is rational.

If $\mathbb{Q}(\chi) \subseteq \mathbb{R}$, say χ is real.

Theorem (Burnside)

A group G has even order if, and only if, some $1_G \neq \chi \in \mathrm{Irr}(G)$ is real.

Remark. Proof is elementary from orthogonality and divisibility of character degrees.

Theorem (Navarro-Tiep '08)

A group G has even order if, and only if, some $1_G \neq \chi \in \mathrm{Irr}(G)$ is rational.

Remark. Unlike the real case, the proof of Navarro-Tiep relies on the Classification of Finite Simple Groups.

R. Gow. If G has even order, can we choose a rational $\mathbf{1}_G \neq \chi \in \mathrm{Irr}(G)$ with odd degree?

Theorem (Navarro-Tiep '08)

G has even order if, and only if, some $\mathbf{1}_G \neq \chi \in \mathrm{Irr}_{2'}(G)$ is rational.

A group G of order divisible by p may not have rational nontrivial characters.

Theorem (Navarro-Tiep '06)

G has order divisible by p if, and only if, $\mathbf{1}_G \neq \chi \in \mathrm{Irr}_{p'}(G)$ has values in $\mathbb{Q}(e^{2\pi i/p})$.

Notice $\mathbb{Q} = \mathbb{Q}(e^{2\pi i/2})$.

Question. Assume $|G|$ is divisible by 2 *or* p, then by Theorem A there is a $\mathbf{1}_G \neq \chi \in \mathrm{Irr}_{\{2,p\}'}(G)$. Can we always find such a χ with values in $\mathbb{Q}$ *or* $\mathbb{Q}(e^{2\pi i/p})$?

Theorem C (Giannelli-Hung-Schaeffer Fry-V. '21)

G group and p prime.

$(|G|, 2p) = 1$ if, and only if, some $\mathbf{1}_G \neq \chi \in \mathrm{Irr}_{\{2,p\}'}(G)$ has values in $\mathbb{Q}(e^{2\pi i/p})$.

Question. Assume now $|G|$ is divisible by 2 *and* p. Can we always find such a χ with values in $\mathbb{Q} \cap \mathbb{Q}(e^{2\pi i/p}) = \mathbb{Q}$?

No! The solvable group A_4 *does* not posses a nontrivial $\{2,3\}'$-degree rational irreducible character.

Open problem. Classify finite groups G admitting a rational $\mathbf{1_G} \neq \chi \in \mathrm{Irr}_{\{2,p\}'}(G)$.

Theorem D (Giannelli-Hung-Schaeffer Fry-V. '21)

G solvable group, p prime and $H \in \mathrm{Hall}_{\{2,p\}}(G)$.

Some $\mathbf{1}_G \neq \chi \in \mathrm{Irr}_{\{2,p\}'}(G)$ is rational if, and only if, H/H' has even order.

Remark. The simple group A_5 *does* posses a nontrivial $\{2,3\}'$-degree rational irreducible character and $\mathrm{A}_4 \in \mathrm{Hall}_{\{2,3\}}(\mathrm{A}_5)$.

3. Principal blocks and two primes

The irreducible characters in the principal p-block of G

$$\mathrm{Irr}(B_p) = \{\chi \in \mathrm{Irr}(G) \mid \sum_{g \in G_{p'}} \chi(g) \neq 0\} \ni \mathbf{1}_G\,,$$

$G_{p'} = \{g \in G \mid (o(g), p) = 1\} \subseteq G$.

For a prime q, write $\mathrm{Irr}_{q'}(B_p) = \mathrm{Irr}(B_p) \cap \mathrm{Irr}_{q'}(G)$.

Theorem (Malle-Navarro '21)

G group, p prime and $P \in \mathrm{Syl_p(G)}$.

$$\mathrm{Irr}_{p'}(B_p) = \mathrm{Irr}(B_p) \text{ if, and only if, } P \text{ is abelian.}$$

Remark. This is the principal block case of Brauer's height zero conjecture!

Problem. Can we characterize $\mathrm{Irr}_{q'}(B_p) = \mathrm{Irr}(B_p)$?

Problem. Can we characterize $\mathrm{Irr}_{q'}(B_p) = \mathrm{Irr}(B_p)$?

Theorem (Navarro-Wolf '01)

G solvable group.

If $\mathrm{Irr}_{q'}(B_p) = \mathrm{Irr}(B_p)$ then some $P \in \mathrm{Syl_p(G)}$ normalizes some $Q \in \mathrm{Syl_q(G)}$.

Remark. The statement above is false in general. For example J_1 with $p = 2$ and $q = 5$.

Restricted Problem. Can one characterize $\mathrm{Irr}_{q'}(B_p) = \mathrm{Irr}(B_p)$ in some cases?

We studied the case where $q = 2$.

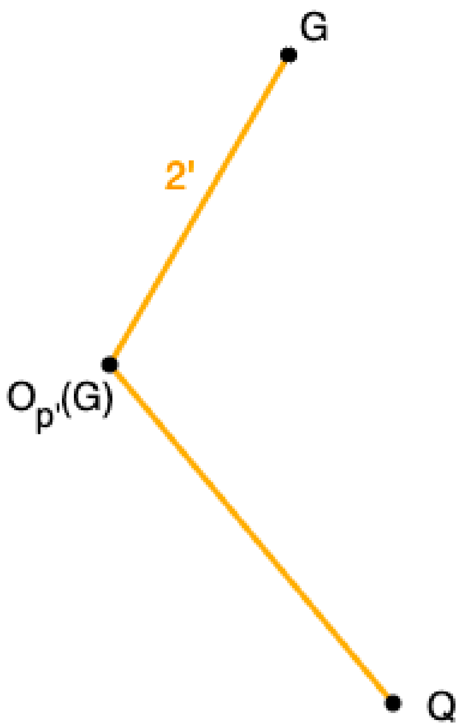
Theorem E (Giannelli-Malle-V. '19)

G group and $p \neq 7$ a prime.

$\mathrm{Irr}_{2'}(B_p) = \mathrm{Irr}(B_p)$ if, and only if, $G/\mathbf{O}_{p'}(G)$ has odd order.

Remark. Under the hypotheses of the above theorem, actually some Sylow p-subgroup of G normalizes a Sylow 2-subgroup of G. (Theorem E extends Navarro-Wolf '01.)

- Take $Q \in \mathrm{Syl}_2(\mathrm{G})$.



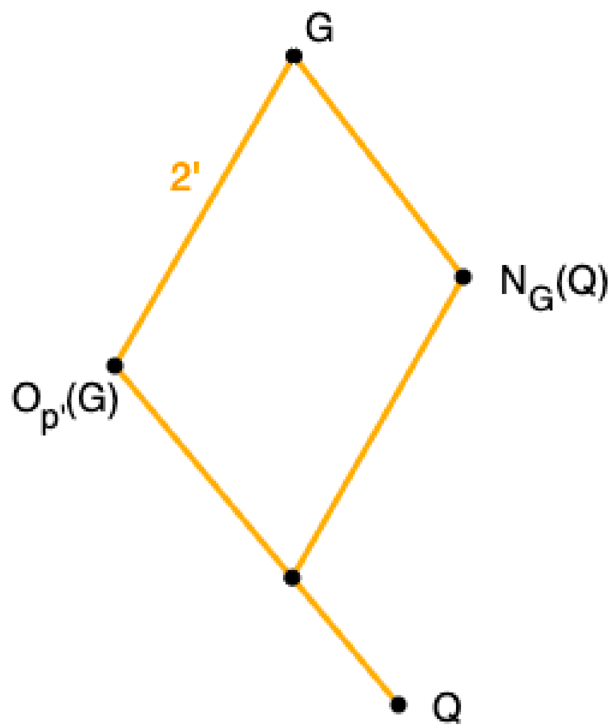
Theorem E (Giannelli-Malle-V. '19)

G group and $p \neq 7$ a prime.

$\mathrm{Irr}_{2'}(B_p) = \mathrm{Irr}(B_p)$ if, and only if, $G/\mathbf{O}_{p'}(G)$ has odd order.

Remark. Under the hypotheses of the above theorem, actually some Sylow p-subgroup of G normalizes a Sylow 2-subgroup of G. (Theorem E extends Navarro-Wolf '01.)

- Take $Q \in \mathrm{Syl}_2(\mathrm{G})$.
- By the Frattini argument $G = \mathbf{O}_{p'}(G)\mathbf{N}_G(Q)$.



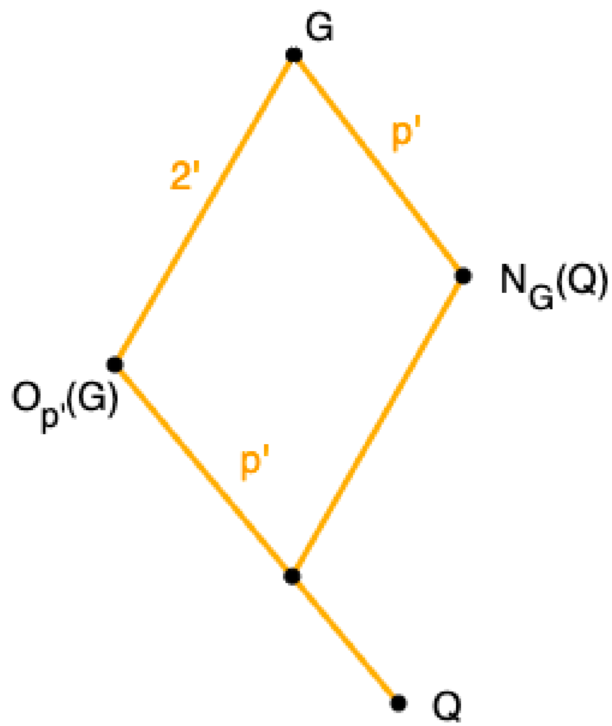
Theorem E (Giannelli-Malle-V. '19)

G group and $p \neq 7$ a prime.

$\mathrm{Irr}_{2'}(B_p) = \mathrm{Irr}(B_p)$ if, and only if, $G/\mathbf{O}_{p'}(G)$ has odd order.

Remark. Under the hypotheses of the above theorem, actually some Sylow p-subgroup of G normalizes a Sylow 2-subgroup of G. (Theorem E extends Navarro-Wolf '01.)

- Take $Q \in \mathrm{Syl}_2(\mathrm{G})$.
- By the Frattini argument $G = \mathbf{O}_{p'}(G)\mathbf{N}_G(Q)$.
- In particular, some $P \in \mathrm{Syl}_\mathrm{p}(\mathrm{G})$ normalizes Q.



Maybe $\mathrm{Irr}_{q'}(B_p) = \mathrm{Irr}(B_p)$ is a too weak condition for two primes!

Conjecture (Malle-Navarro '20)

$\mathrm{Irr}_{q'}(B_p) = \mathrm{Irr}(B_p)$ and $\mathrm{Irr}_{p'}(B_q) = \mathrm{Irr}(B_q)$ if, and only if, $P \in \mathrm{Syl_p(G)}$ and $Q \in \mathrm{Syl_q(G)}$ centralize each other.

- If $p = q$ then it is the principal block case of Brauer's height zero conjecture.

($\Leftarrow$) Proven by Malle and Navarro.
($\Rightarrow$) Reduced to show that the principal blocks of simple non-abelian groups satisfy the inductive Alperin-McKay condition.

Theorem (Giannelli-Malle-V. '19, Giannelli-Meini '20)

Given two primes p and q, and B_p the principal p-block of A_n $(n \geqslant 5)$. There is some $\chi \in (B_p)$ of degree divisible by q.
In particular, alternating groups satisfy the Malle-Navarro conjecture on principal blocks and two primes.

Conjecture (Malle-Navarro '20)

$\mathrm{Irr}_{q'}(B_p) = \mathrm{Irr}(B_p)$ and $\mathrm{Irr}_{p'}(B_p) = \mathrm{Irr}(B_p)$ if, and only if, some $P \in \mathrm{Syl_p(G)}$ and $Q \in \mathrm{Syl_q(G)}$ centralize each other.

This conjecture characterizes the existence of nilpotent Hall $\{p, q\}$-subgroups.

Another recent conjecture on this topic is the following.

Conjecture (Liu-Willems-Xiong-Zhang '20)

If $\mathrm{Irr}(B_p) \cap \mathrm{Irr}(B_p) = \{\mathbf{1}_G\}$ then G has nilpotent Hall $\{p, q\}$-subgroups.

Remark. The authors prove their conjecture holds in general if it holds for almost simple groups.

Thanks for your attention!